GAUSSIAN SPREADING OF WAVES FROM A SPHERICAL RADIATING SOURCE

By

A. O. AKALA*, A. B ADELOYE* and E.O. SOMOYE**

*Department of Physics, University of Lagos, Akoka, Yaba, Lagos

**Department of Physics, Lagos State University, Ojo, Lagos.

ABSTRACT

Improving antenna performance in response to challenging environmental behaviour (stochastic medium) requires urgent call for scientific attention. The present research looked into this by constructing, and numerically solving Green's functions. The Green's function solution follows Improving antenna performance in response to challenging environmental behaviour (stochastic medium) requires urgent call for scientific asymptotic behaviour, with a marked decrease in the strength of the electromagnetic field as the radial distance increases. This is due to the transfer of energy via spherical wave front (i.e energy is dissipated as the wave radiates away from the source). Besides, we interpreted the slow decay in the field at large times to be connected with the gradual relaxation of the medium after the initial wave front passed. Overall, this effort has practical application in designing omni-directional radiating system (antenna), especially when there is need to address the challenge of aperture (arraya0 directivity.

Keywords: Electromagnetic wave, source aperture, radial distance, Green's function.

NOTATIONS

α_o ≡ Attenuation coefficient

U(t) ≡ Heaviside step function

$G(R,t)$ ≡ Three dimensional Green's function

(R/c_o) ≡ Delay term

$(1/4\pi R)$ ≡ Spherical diffraction factor

c_z ≡ Phase speed in the limit of zero frequency

c_∞ ≡ Phase speed in the limit of infinitely high frequency

1.0 INTRODUCTION

In recent times, the need to intensify efforts on improving steering mechanism that will enhance antenna directivity performance in such a way that gain will not be unnecessarily sacrificed has been well orchestrated (Brooker, 1991; Brown et al., 1998; Kraus and Markeka, 2002). Critical study of the impacts that the transient wave from radiating systems will suffer as it propagates from the source (through a stochastic medium) to the receiver has been prescribed (Brown et al., 1998). One notable mathematical tool that can be used to characterize the distribution of electromagnetic wave from source to receiver is the Green's function. To this end, we use the construction, and numerical computations of Green's function to analyze the Gaussian spreading of waves from a spherical radiating source with a view to addressing the above mentioned challenge. The observed Green's function follows asymptotic behaviour with rapid decay in the amplitude.

There is marked decrease in the strength of the electromagnetic field as the radial distance increases. This is due to the transfer of energy from spherical wave front to a slowly decaying wave (i.e. energy is dissipated as the wave radiates away from the source (Jackson, 1974)). Besides, the slow decay is interpreted to be connected with the gradual relaxation of the medium after the initial wave front has passed. This paper has practical applications in designing omni-directional radiating system (antenna), especially when the need to address the challenge of aperture (array) directivity arises.

Section 2.0 presents the theory of Green's function. The numerical scheme and the results are discussed in sections 3.0 and 4.0 respectively, followed by the conclusion in section 5.0, and then the Appendix.

2.0 THEORY

The Szabo wave equation (Szabo, 1994) approximates power law media with attenuation coefficient given by;

$$\alpha(\omega) = \alpha_o |\omega|^y \qquad (1)$$

Leibler et al. (2004) expressed this equation in terms of fractional derivatives, whereby the machinery of fractional calculus was utilized. For $y \neq 1$, Equation (1) becomes;

$$\nabla^2 p - \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_o}{c_o \cos(\pi y/2)}\frac{\partial^{y+1} p}{\partial t^{y+1}} = 0 \quad (2)$$

where the third term accounts for dispersive loss. In special cases of $y = 0$ and $y = 2$, the fractional derivative term reduces to first and third temporal derivatives respectively, which can then be solved by standard methods.

The $y = 0$ case corresponds to the telegrapher's equation (Kelly et al., 2008).

$$\nabla^2 p - \frac{1}{c_\infty^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_o}{c_\infty}\frac{\partial p}{\partial t} = 0 \qquad (3)$$

Equation (3) can be used to model one dimensional damped string motion and electromagnetic wave propagation in conductive media. The reference frequency c_∞ is the phase speed in the limit of infinitely high frequency and α_o is the attenuation coefficient for $\omega = 1$. The three dimensional Green's function for Equation (3) consists of two terms: an exponential attenuated spherical wave, which is localized at time R/c_∞, and a wake involving a modified Bessel function, which extends for infinite time. For small α_o, the three dimensional Green's function can be expressed as;

$$G(R,t) \approx U(t)\exp^{-\alpha_o R}\frac{\delta(t - R/c_\infty)}{4\pi R} \qquad (4)$$

where U(t) is the Heaviside step function. Similarly, the $y = 2$ case corresponds to the Blackstock equation (Kelly et al., 2008);

$$\nabla^2 p - \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_o}{c_o}\frac{\partial^3 p}{\partial t^3} = 0 \qquad (5)$$

Equation (5) can be used to model the acoustic wave propagation in viscous media under plane wave approximation for small α_o, and small frequency c_o (the phase speed in the limit of zero frequency).

Using Fourier transformation technique, three dimensional Green's function of Eq. (5) can be written as;

$$G(R,t) \approx \frac{1}{4\pi R}\frac{1}{\sqrt{4\pi R\alpha_o}}\exp\left(-\frac{(t - R/c)^2}{4R\alpha_o}\right) \qquad (6)$$

Equation (6) predicts Gaussian spreading of spherical waves as the field radiates away from the source (Kelly and McGough, 2008). The details of the derivations of Equation (6) are shown in Appendix A.

3.0 NUMERICAL SCHEME

This paper centres on the Blackstock component of Szabo equation, whereby the Heaviside step function is set to unity. The numerical framework is based on Equation (6). This equation predicts the Gaussian spreading of spherical waves as the field radiates away from the source, using the theory of Green's conditions. The final Green's function is characterized with delay term, (R/c_o) scaled by and multiplied by $(2\alpha_o R)$ spherical diffraction factor (Kelly et al., 2008).

The parameters for the simulations are arbitrary. The spherical source (radiator) is assumed to have an aperture size $a = 0.01m$. The attenuation coefficient is assumed to maintain constant value of 0.1 throughout the analysis, and small phase speed was also considered. We considered simulation time from 0 to 80μs for three different radial distances; 0.01m, 0.1m and 1m respectively. The radial distance 0.1m (R>a) is considered as the intermediate zone for the field distribution, while that of the 1m (R>>a) is considered as the far zone. The computational results are then plotted in Figures 1-3 respectively. The plots are thereafter superimposed in Figure 4. The idea is to observe the symmetric behaviour of the fields at varying radial distances in time domain.

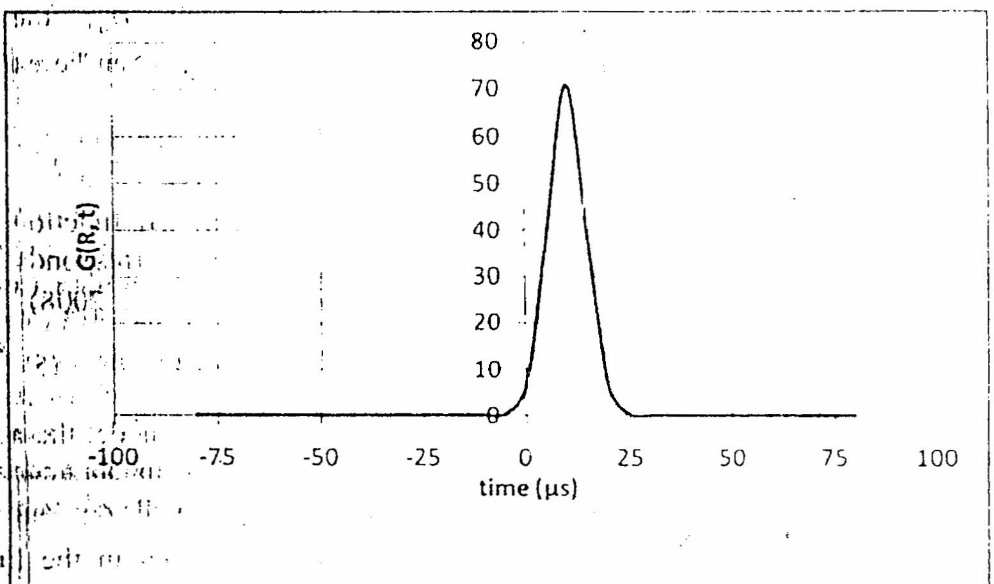

Figure 1: Green's Function Versus Propagation Time for 0.01m Radial Distance.

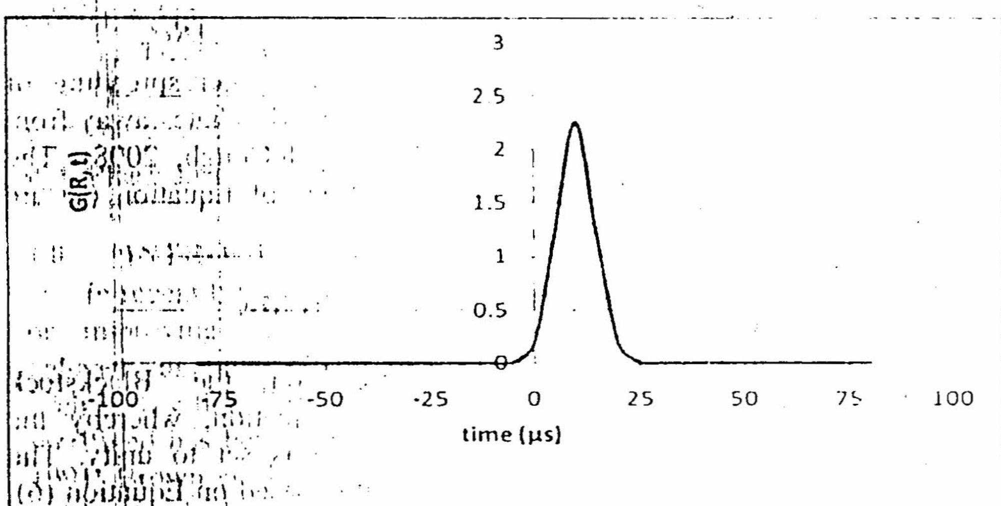

Figure 2: Green's Function Versus Propagation Time for 0.1m Radial Distance.

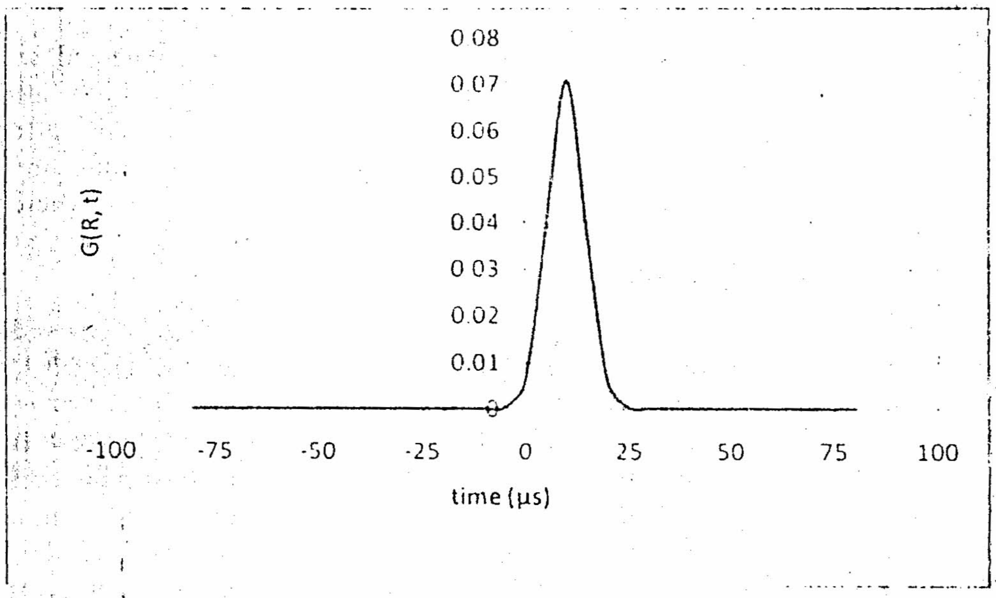

Figure 3: Green's Function Versus Propagation Time for 1m Radial Distance.

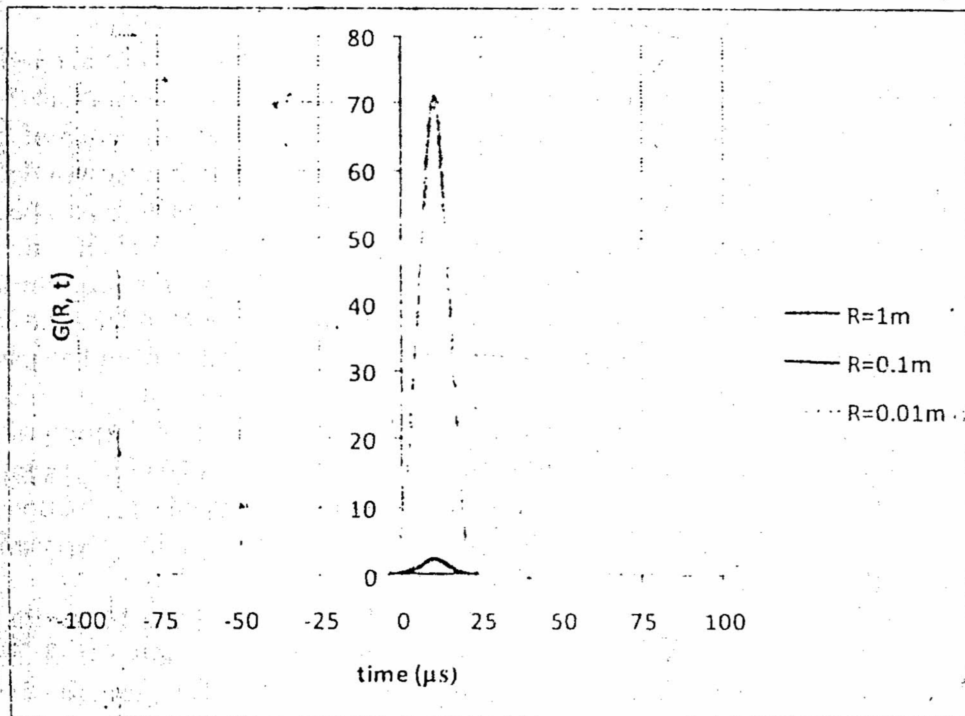

Figure 4: Superimposition of the Green's Function Versus Propagation Time for the Three Radial Distances.

4.0 DISCUSSION

The thrust of this paper centres on the construction and numerical computation of Green's function in time domain, with a view to analysing the status of the electromagnetic wave as it propagates from the source (spherical radiator) to a defined radial distance, R. The Green's function (Equation (6)) predicts the Gaussian spreading of spherical waves as the field radiates away from the source. Equation (6) is localized at $t = (R/c_o)$.

The parameterization of the simulation is purely arbitrary. The attenuation coefficient is assumed to be small for numerical convenience. Under practical conditions, the attenuation coefficient is environmentally dependent. The numerical computation for the Green's function is carried out for varying propagation time for three different radial distances (0.01m: near zone, 0.1m: intermediate zone, and 1m: far zone), having assumed that the source has a spherical aperture size of 0.01m (radius).

The numerical results are plotted in Figures 1-3, and comparative observation in Fig. 4. The observed Green's function follows Gaussian distribution (Kreyszig, 2005; Akala et al., 2010), with common symmetry about t=10μs and the solution falls asymp-totically. The Green's function decays rapidly from the point of symmetry to about 20μs. As the time approaches infinity, the Green's function on the other hand approaches zero.

A decrease in the tail of the amplitude of the Green's function is noted as the radial distance from the source increases. We interpreted this to have a direct relationship with the strength of the electromagnetic field. In other words, the Green's function represents the strength of the electromagnetic field. In comparison with the aperture size, there is marked decrease in the strength of the electromagnetic field as the radial distance increases. This was observed to decrease from about 70 (unit) to 0.07 (unit) for the near zone and far zone respectively.

The physical interpretation of the fall in the strength of the electromagnetic field is that energy is being transferred from spherical wave front to a slowly decaying wave (i.e. energy is dissipated as the wave radiates away from the source (Jackson, 1974)). Besides, the slowly decay is interpreted to be connected with the gradual relaxation of the medium after the initial wave front has passed. This effort has practical application in omni-directional radiating system (antenna) designs, whereby the challenge of aperture (array) directivity needs to be addressed.

5.0 CONCLUSION

The Gaussian spreading of waves from a spherical radiating source has been presented, with the construction and numerical computation of Green's function as the centre tool. The observed Green's function follows asymptotic behaviour with rapid decay in the amplitude of the Green's function from the centre of symmetry in time to about 20μs. The function also approaches zero as the time approaches infinity, which implies that the Green's function as applies to electromagnetic radiation obeys Gaussian spreading.

There is marked decrease in the strength of the electromagnetic field as the radial distance increases. This is due to the transfer of energy from spherical wave front to a slowly decaying wave (i.e. energy is dissipated as the wave radiates away from the source (Jackson, 1974). Besides, the slowly decay is interpreted to be connected with the gradual relaxation of the medium after the initial wave front has passed.

Overall, the effort presented in this paper has practical applications in the design of omni-directional radiating system (antenna), especially when the need to address the challenge of aperture (array) directivity arises.

ACKNOWLEDGEMENTS

The first author acknowledges all the past discussions on classical electro-dynamics with Dr. J. Akinrimisi (late) with unreserved gratitude.

$$G(R,t) = \frac{1}{8\pi^3}\int_0^\infty\int_0^\pi\int_0^{2\pi} G_p(t)\exp(ipR\cos\theta_p)\times\sin\theta_p\, p^2\, d\varphi_p\, d\theta_p\, dp \quad (A6)$$

After evaluating φ_p and θ_p integrals in equation (A6),

$$G\,(R,t) = \frac{1}{2\pi^2 R}\int_0^\infty p\sin(pR)G_p(t)dp \quad (A7)$$

Substituting equation (A5) into equation (A7) and exploiting the evenness of the integrand yields;

$$G\,(R,t) = \frac{c_o U(t)}{4\pi^2 R}\int_{-\infty}^{\infty}\frac{\sin(pR)\exp\left(-\frac{\gamma c_o^2 p^2 t}{2}\right)\sin(c_o p\chi t)}{\chi}dp \quad (A8)$$

It is worthy to note that equation (A8) is an exact Fourier integral representation of the three dimensional Green's function. Evaluating this integral in close form seems impossible. For simplicity, there is need to set χ to unity, so that,

$$G\,(R,t) = \frac{c_o U(t)}{4\pi^2 R}\int_{-\infty}^{\infty}\sin(pR)\exp\left(-\frac{\gamma c_o^2 p^2 t}{2}\right)\sin(c_o pt)dp \quad (A9)$$

Adopting an identity based on cosine addition formula,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\exp(-ap^2)\sin(pR)\sin(bp)dp = \frac{1}{\sqrt{4\pi a}}\left[\exp\left(-(R-b)^2/4a\right)-\exp\left(-(R+b)^2/4a\right)\right] \quad (A10)$$

Using the identity in equation (A10), and taking $a = \gamma c_o^2 t/2$ and $b = c_o t$; equation (A9) can be expressed as;

$$G(R,t) \approx \frac{U(t)}{4\pi R\sqrt{2\pi\gamma t}}\left[\exp\left(-(t-R/c_o)^2/2\gamma t\right)-\exp\left(-(t+R/c_o)^2/2\gamma t\right)\right] \quad (A11)$$

Setting $\gamma t = 2R\alpha_o$, equation (A11) becomes:

$$G(R,t) \approx \frac{U(t)}{4\pi R}\frac{1}{\sqrt{4\pi R\alpha_o}}\left[\exp\left(-(t-R/c_o)^2/4R\alpha_o\right)-\exp\left(-(t+R/c_o)^2/4R\alpha_o\right)\right] \quad (A12)$$

The first term of equation (A12) represents the outgoing wave, while the second term represents the incoming wave. At large times, the incoming wave term can be neglected.

REFERENCES

(1) Akala, A. O., Oyeyemi, E. O., Somoye, E. O., Adeloye, A. B. and Adewale, A. O (2010). Variability of foF2 in the African equatorial ionosphere, *Adv. Space Res.*, doi:10.1016/j.asr.2010.01.003, in press.

(2) Brooker, E (1991). Practical Phased Array Antenna Systems, Artech House Inc, MA, USA.

(3) Brown, A., E. Holm, and K. M. Grove (1998): GPS Ionospheric Scintillation Measurements Using a Beam Steering Antenna Array for Improved Signal/Noise, *Proc. ION-NTM.*

(4) Jackson J. D. (1974): Classical Electrodynamics, 2nd edn., John Wiley & Sons, NY, U.S.A.

(5) Kelly, J. F. and R. J. McGough (2008): Causal Impulse Response for Circular Sources in Viscous Media, *J. Acoust. Soc. Am.* **123**(4), pp. 2107-2116.

(6) Kelly, J. F., R. J. McGough, and M. M. Meerschaert (2008): Analytical Time Domain Green's Functions for Power-Law Media, *J. Acoust. Soc. Am.* **124**(5), pp. 2861-2872.

(7) Kraus J. D. and R. J. Marketka (2002). Antenna for all Applications, Mc GrawHill Inc, 3rd Edn, NY, USA.

(8) Kreyszig, E. (2005): Advanced Engineering Mathematics, 8th edn., John Wiley & Sons (Asia) Pte Ltd., Singapore.

(9) Leibler, M., S. Ginter, T. Dreyer, and R. E. Riedlinger (2004): Full Wave Modelling of Therapeutic Ultrasound: Efficient Time-Domain Implementation of the Frequency Power Law Attenuation, *J. Acoust. Soc. Am.* **116**, pp. 2742-2750.

(10) Szabo, T. L. (1994): Time Domain Wave Equations for Lossy Media Obeying a Frequency Power Law, *J. Acoust. Soc. Am.* **96**, pp. 491-500.

APPENDIX A

Defining the Green's function that satisfies the Stokes equation in space-time domain as;

$$\nabla^2 G - \frac{1}{c_o^2}\frac{\partial^2 G}{\partial t^2} - \gamma\frac{\partial \nabla^2 G}{\partial t} = -\delta(t)\delta(R) \quad \text{(A1)}$$

where c_o is the thermodynamic speed of sound and γ is the relaxation time of the medium. Using Fourier transforms technique, Kelly and McGough (2008) expressed equation (1) as;

$$\left(-p^2 + \frac{\omega^2}{c_o^2} - \gamma i p^2 \omega\right)G_{\omega p} = -1 \quad \text{(A2)}$$

where $p = |\mathbf{p}|$ is the magnitude of the spatial frequency vector **p**.

Solving for $G_{\omega p}$ and performing an inverse Fourier transform over ω gives,

$$G_p = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{\omega p}\exp(i\omega t)d\omega \quad \text{(A3)}$$

$$= -\frac{c_o^2}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(i\omega t)}{(\omega-\omega_+)(\omega-\omega_-)}d\omega \quad \text{(A4)}$$

where $\omega_\pm = i\gamma c_o^2 p^2/2 \pm c_o p\chi$ and $\chi = \sqrt{1-\gamma^2 c_o^2 p^2/4}$. Evaluating equation (A4) via contour integration yields;

$$G_p(t) = \frac{c_o U(t)}{\chi p}\exp\left(-\frac{\gamma c_o^2 p^2 t}{2}\right)\sin(c_o p\chi t) \quad \text{(A5)}$$

Employing three-fold inverse Fourier transform, the three dimensional Green's function (spherical coordinate system) can be recovered as,